Capacity of Fading Channels

In Shannon theory, a discrete memory-less channel (DMC) is characterized by a conditional probability distribution, $p_{X|S}(\cdot)$, where $s$ represents the input to the channel (the transmitted signal/symbol) and $x$ represents the output of the channel (the received signal/symbol).

According to Shannon’s channel coding theorem, associated with any DMC, $p_{X|S}(\cdot)$, and any family of admissible input distributions, $S$, there exists a quantity referred to as channel capacity:

$$C = \max_{p_S(\cdot) \in S} E \log \frac{p(X|S)}{p(X)},$$  \hspace{1cm} (1)

such that arbitrarily reliable communication can be achieved for all rates $R < C$. Moreover, for rates $R > C$ the probability of error is strictly bounded above zero. (When the $\log$ in the above formula is in base 2, capacity is in bits/channel use. When the $\log$ is in base $e$, capacity is in nats/channel use. We will often use the latter for convenience.)

We further comment that the quantity

$$I(X; S) = E \log \frac{p(X|S)}{p(X)},$$

is referred to as mutual information and physically represents the amount of information that can be deduced about $s$, based on observing $x$ (equivalently, the amount of information that can be deduced about $x$, based on observing $s$). Capacity therefore is essentially the maximum, over the input distribution of $s$, of the amount of information about $s$ that can be inferred from $x$.

Channel capacity represents the fundamental limitation for information transmission over any communications channel.
The additive white Gaussian noise (AWGN) channel

As an exercise, we will look at the AWGN channel, which has the following model

\[ x = s + v, \]  

(2)

where \( v \) is a zero-mean, \( \sigma^2 \)-variance, circularly-symmetric complex Gaussian random variable. Circularly-symmetric refers to the fact that the real and imaginary parts of \( v = v_R + jv_I \) are independent, zero-mean, \( \frac{\sigma^2}{2} \)-variance, real-valued Gaussian random variables. Therefore it is straightforward to see that

\[ p_V(v) = \frac{e^{-|v|^2}}{\pi\sigma^2}. \]

Equation (2) clearly implies that the conditional probability distribution for the AWGN channel is

\[ p_X|S(x, s) = \frac{e^{-|x-s|^2}}{\pi\sigma^2}. \]

To compute the capacity for the AWGN channel we need to say something about the transmitted signal \( s \). We will assume \( s \) to be a complex-valued signal with unit power, i.e., \( E|s|^2 = 1 \). This implies that \( \mathcal{S} \) is the family of input distributions with the property that \( \int |s|^2 p_S(s) ds = 1 \).

Now

\[ E \log p(X|S) = \log \frac{1}{\pi\sigma^2} - \frac{E|x-s|^2}{\sigma^2} = \log \frac{1}{\pi\sigma^2} - 1 = \log \frac{1}{\pi\sigma^2} e, \]

which is independent of the distribution of \( s \) and only depends on the fact that, conditioned on \( s \), \( x \) is a \( \sigma^2 \)-variance Gaussian random variable with mean \( s \). Therefore to obtain capacity we need to maximize

\[ E \log \frac{1}{p(X)}, \]

over the input distribution. Now note that \( x \) is a random variable with variance \( E|x|^2 = 1 + \sigma^2 \). Using a Lagrange multiplier technique it is straightforward to see that the density that maximizes \( E \log \frac{1}{p(X)} \), subject to the variance constraint, \( |x|^2 = 1 + \sigma^2 \), is the Gaussian distribution

\[ p(X) = \frac{e^{-|x|^2/(1+\sigma^2)}}{\pi(1+\sigma^2)}. \]
Note that the above distribution on $X$ can be obtained when $S$ is taken to have the distribution $p(S) = \frac{e^{-|S|^2/\sigma^2}}{\pi \sigma^2}$. In this case, we obtain $E \log \frac{1}{\mathcal{N}(\mu)} = \log \frac{1}{\pi(1+\sigma^2)e}$, and so

$$C = \log \left(1 + \frac{1}{\sigma^2}\right) = \log(1 + \text{SNR}),$$

where $1/\sigma^2$ is clearly the SNR.

Finally, we reiterate that the capacity-achieving distribution on $S$ is Gaussian.

**Narrow-band Fading Channels**

Narrow-band fading channels can be described by the following equation

$$x = hs + v,$$

where the additive noise $v$ is a zero-mean, $\sigma^2$-variance, circularly-symmetric complex Gaussian random variable. The channel $h$ is also a Gaussian random variable, independent of $v$. For Rayleigh fading $h$ is zero-mean, whereas for Rician fading it has a nonzero mean. In either case, we will assume that $E|h|^2 = 1$. Coupled with the input power constraint, $E|s|^2 = 1$, this implies that the SNR is given by $1/\sigma^2$.

Note that the only difference between a fading channel and the AWGN channel is the random channel gain $h$. We will now study the effect of this random gain on the channel capacity.

**Receiver side information**

A standard assumption in many wireless systems is that the receiver knows the channel gain, $h$. This is referred to as channel side information at the receiver and is a reasonable assumption when the channel is fading slowly. The reason is that training symbols (or pilot signals) may be used to “learn” the channel, after which the channel can be considered known at the receiver, since its value changes slowly with time.

In any event, in this case the receiver has two known quantities, $h$ and $x$ and so the mutual information under consideration is given by

$$I((X, H); S) = E \log \frac{p(X, H|S)}{P(X, H)}.$$ 

Some simple algebra shows that

$$I((X, H); S) = I(X; S|H) + I(H; S).$$
(Show this!) But since $H$ and $S$ are independent (the fading has nothing to do with the transmitted signal), we have $I(H;S) = 0$, and so the capacity of the fading channel with receiver side information is given by

$$C = \max_{p_S(\cdot), E|s|^2 = 1} I(X; S | H).$$

Note that, apart from the conditioning on $H$, this is exactly the same mutual information that is considered for the AWGN channel. Since the capacity-achieving distribution on $s$ does not depend on $h$, conditioned on $h$, the capacity is given by $\log(1 + |h|^2/\sigma^2)$. Therefore the capacity of the fading channel with receiver side information is given by

$$E \log \left( 1 + \frac{|h|^2}{\sigma^2} \right) = E \log \left( 1 + |h|^2 \text{SNR} \right). \quad (5)$$

Note that, in view of Jensen’s inequality ($E \log(\cdot) \leq \log E(\cdot)$), the capacity of a fading channel is always less than the capacity of an AWGN channel at the same SNR. Therefore fading hurts the capacity.

**Transmitter side information**

Another assumption in wireless systems is that the transmitter knows the channel gain, $h$. This is referred to as channel side information at the transmitter and is a reasonable assumption when the channel is fading slowly and when there exists a reliable feedback (or reverse link) channel that allows the receiver to feed back the channel gain to the transmitter.

In this case the transmitter can adjust its power level according to the current value of $h$, so that we may write

$$x = \sqrt{t(h)h} s + v,$$

where $\sqrt{t(h)}$ represents the instantaneous transmit power which depends on $h$. Clearly, the power constraint implies that

$$E t = \int t(h)p(h) dh = 1.$$

Therefore the capacity becomes

$$C = \max_{t(\cdot),E t = 1} E \log \left( 1 + t(h) \frac{|h|^2}{\sigma^2} \right). \quad (6)$$
Using Lagrange multipliers, the power adaptation satisfies the following so-called time water-filling policy:

\[ t(h) = \begin{cases} \frac{\lambda}{|h|^2} - \frac{\lambda^2}{|h|^4} & \text{if } \frac{|h|^2}{\sigma^2} \geq \lambda \\ 0 & \text{if } \frac{|h|^2}{\sigma^2} < \lambda \end{cases} \] \quad (7)

where \( \lambda \) satisfies the equation

\[ \int_{\sigma^2\lambda}^{\infty} \left( \frac{1}{\lambda} - \frac{\sigma^2}{|h|^2} \right) p(|h|^2) \, |h|^2 \, dh. \] \quad (8)

Replacing the optimal policy into the capacity expression yields

\[ C = \int_{\sigma^2\lambda}^{\infty} \log \frac{|h|^2}{\sigma^2\lambda} p(|h|^2) \, |h|^2 \, dh. \] \quad (9)

Note that the water-filling policy transmits at higher powers when the channel fading gain is favorable. On the other hand if the fading magnitude is below a certain threshold, the optimal transmit power is zero.

**Channel inversion**

A scheme that is often used is channel inversion. In this case, the transmit power is given by

\[ \sqrt{t(h)} = \frac{K}{h}, \]

where \( K \) is a constant chosen so that the power constraint is satisfied. Clearly,

\[ K^2 = \frac{1}{E[|h|^2]} . \]

Note that with channel inversion, the receiver observes the AWGN channel

\[ x = Ks + v, \]

so that the capacity is

\[ C = \log \left( 1 + \frac{K^2}{\sigma^2} \right). \]

By virtue of Jensen’s inequality, \( K^2 \leq 1 \), and so the above is less than the AWGN channel capacity. In fact, for Rayleigh fading it follows that the capacity of channel inversion is zero!
Channel Unknown to Everyone - No Side Information

- In reality, the wireless fading gain $h$ is never exactly known to the receiver or transmitter.

- Therefore the most interesting capacity to compute is that of an unknown fading channel.

- Here the capacity depends not only on the marginal density of the channel (Rayleigh, Rician, etc.), as it did in the cases of receiver and transmitter side information, but also on the temporal correlation of the channel.

- In fact, the channel must now be written as

$$x_i = h_is_i + v_i,$$

where $\{h_i\}$ and $\{v_i\}$ are Gaussian random processes. The additive noise $\{v_i\}$ is assumed to be temporally white, and so the channel is essentially determined by the first- and second-order statistics (mean and autocorrelation functions) of the random process $\{h_i\}$. Some standard assumptions for this autocorrelation function includes the Bessel function model

$$R_h(i) = J_0(2\pi f_DT_i),$$

where $f_D$ is the Doppler frequency and $T_i$ is the symbol rate, which holds for isotropic scattering, as well as the first order Gauss-Markov model

$$R_h(i) = \alpha|i|,$$

which is often more amenable to analysis.

- In either case, computing capacity is an open problem.