On Robust Two-Block Problems

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Abstract

In this paper we consider the following robust two-block problem that arises in estimation and in full-information control: minimize the worst-case $H^\infty$ norm of a two-block transfer matrix whose elements contain $H^\infty$-norm-bounded modeling errors. We show that, when the underlying systems are single-input/single-output, and if the modeling errors are "small enough", then the robust two-block problem can be solved by solving a one-dimensional family of appropriately-weighted "modeling-error-free" two-block problems. We also study the consequences of this result to a robust tracking problem, where the optimal solution can be explicitly found.

1 Introduction

In this paper we study the robust two-block problem

$$\inf_{Q(\cdot) \in H^\infty} \sup_{(\Delta P_1(\cdot), \Delta P_2(\cdot)) \in B_\delta^\infty} \|T_Q(P_1 + \delta P_1, P_2 + \delta P_1)\|_{\infty},$$

(1)

where

$$T_Q(P_1 + \delta P_1, P_2 + \delta P_2) = \left[ \begin{array}{c} P_1(z) + \Delta P_1(z) + (P_2(z) + \Delta P_2(z))Q(z) \\ Q(z) \end{array} \right],$$

and

$$B_\delta^\infty = \{ (\Delta P_1(\cdot), \Delta P_2(\cdot)), \| [ \Delta P_1(\cdot) \quad \Delta P_2(\cdot) ] \|_{\infty} \leq \delta \},$$

with $Q(\cdot)$, $P_1(\cdot)$, $P_2(\cdot)$, $\Delta P_1(\cdot)$ and $\Delta P_2(\cdot)$ all scalar functions in $H^\infty$. This problem can be considered a robust version of the standard two-block problem

$$\inf_{Q(\cdot) \in H^\infty} \| [ P_1(z) + P_2(z)Q(z) ] \|_{\infty},$$

(2)

that arises in $H^\infty$ full-information control, and in $H^\infty$ estimation, which further allows one to consider possible modeling errors $\Delta P_1(\cdot)$ and $\Delta P_2(\cdot)$ for the nominal plants $P_1(\cdot)$ and $P_2(\cdot)$. In problem (1) both the objective and the modeling errors are measured in the $H^\infty$ norm. Thus $\delta > 0$ is a measure of the modeling error allowed for in (1).

2 Main Result

Problem (1) is, of course, a highly nonlinear problem and satisfactory solutions to date do not exist. In this paper, we show that for "small enough" modeling errors problem (1) can be solved by doing a one-dimensional search over the solution of a certain family of weighted standard two-block problems.

Thus, consider the solution to the following weighted two-block problem:

$$f(\epsilon) \triangleq \inf_{Q(\cdot) \in H^\infty} \left\{ \sqrt{1 + \frac{1}{\epsilon^2}} \left( \frac{P_1(z) + P_2(z)Q(z)}{\sqrt{1 + (1 + \frac{1}{\epsilon^2})\delta^2Q(z)}} \right)^2 \right\}^2$$

$$+ (1 + \frac{1}{\epsilon^2})\delta^2, \quad \epsilon > 0.$$  

(3)

The above problem can be readily solved for any value of $\epsilon$ (say, by using Riccati-based techniques when $P_1(\cdot)$ and $P_2(\cdot)$ are rational), and so $f(\epsilon)$ is easy to compute. Moreover, it can be shown that $f(\cdot)$ is, in general, a nonconvex continuous function of $\epsilon$. Suppose now that we perform a one-dimensional search over $\epsilon > 0$, and determine

$$\epsilon^* \triangleq \inf_{\epsilon > 0} f(\epsilon).$$

(4)

Then we have the following result.

Theorem 1 (Robust Two-Block Problem)
There exists a $\hat{\delta} > 0$, such that for all $\delta < \hat{\delta}$, the solution to the robust two-block problem (1) can be found from the solution to the weighted two-block problem

$$\inf_{Q(\cdot) \in H^\infty} \left\{ \sqrt{1 + \frac{1}{\epsilon^2}} \left( \frac{P_1(z) + P_2(z)Q(z)}{\sqrt{1 + (1 + \frac{1}{\epsilon^2})\delta^2Q(z)}} \right)^2 \right\}^2$$

$$(1 + \frac{1}{\epsilon^2})\delta^2.$$  

(5)
The above theorem states that if the modeling error is less than $\delta > 0$, then the robust two-block problem (1) can be solved using a one-dimensional search over a family of weighted two-block problems. Moreover, the solution to (1) is the same as the solution to a certain weighted two-block problem, with optimal weighting determined by $\epsilon^*$. This has rather interesting physical implications since it states that the modeling errors $\Delta P_1(\cdot)$ and $\Delta P_2(\cdot)$ can be dealt with by appropriately weighting the modeling-error-free two-block problem (2).

Of course, this result raises several issues:

- How does Theorem 1 generalize to matrix plants?
- How does Theorem 1 generalize to four-block problems?
- How large is the value of $\delta$ in Theorem 1?

Currently all three questions are open. To gain some insight into the third question, let us consider the robust tracking problem.

### 3 Robust Tracking

The $H^\infty$ tracking problem corresponds to $P_1(\cdot) = 1$ and $P_2(\cdot) = -P(\cdot)$, so that the robust tracking problem takes the form

$$\inf_{Q(\cdot) \in H^\infty} \sup_{\|\Delta P(\cdot)\|_\infty \leq \delta} \left\| \left[ 1 - \left( P(\cdot) + \Delta P(\cdot) \right) Q(\cdot) \right] / Q(\cdot) \right\|_\infty.$$  

(6)

The modeling-error-free tracking problem,

$$\inf_{Q(\cdot) \in H^\infty} \left\| \left[ 1 - P(\cdot) Q(\cdot) \right] / Q(\cdot) \right\|_\infty \overset{\Delta}{=} \gamma_{\text{opt}},$$

(7)

has been studied in [1], where it is shown:

- If $P(\cdot)$ is minimum phase, then
  $$\gamma_{\text{opt}} = \frac{1}{1 + \min_{\omega \in [0, 2\pi]} |P(e^{j\omega})|^2}.$$  

(8)

- If $P(\cdot)$ is minimum phase, then
  $$\gamma_{\text{opt}} = 1.$$  

(9)

Let us now return to problem (6). Clearly, if $P(\cdot)$ is nonminimum phase, we can always obtain an $H^\infty$ norm of unity in the objective cost by setting $Q(\cdot) = 0$. This is the same value obtained in the modeling-error-free case. Therefore, let us focus on the case where $P(\cdot)$ is minimum phase.

Here we will have to distinguish between two cases:

(i) $\delta \geq \min_{\omega \in [0, 2\pi]} |P(e^{j\omega})|^2 \overset{\Delta}{=} p_{\text{min}}$. In this case, there exist modeling errors for which $P(\cdot) + \Delta P(\cdot)$ is non-minimum phase. Thus here the best choice is $Q(\cdot) = 0$, which results in an $H^\infty$ norm of unity.

(ii) $\delta < \min_{\omega \in [0, 2\pi]} |P(e^{j\omega})|^2 \overset{\Delta}{=} p_{\text{min}}$. In this case, $P(\cdot) + \Delta P(\cdot)$ is always minimum phase.

Thus, clearly the case of interest is case (ii), above. The next result shows that for this case, the optimally-weighted two-block problem always solves (6).

**Theorem 2 (Robust Tracking)** Consider problem (6) and suppose that

$$\delta < \min_{\omega \in [0, 2\pi]} |P(e^{j\omega})|^2 \overset{\Delta}{=} p_{\text{min}}.$$  

Then the solution to problem (6) is given by the solution to the problem,

$$\inf_{Q(\cdot) \in H^\infty} \left\| \frac{\sqrt{1 + \epsilon^* \left( 1 - P(\cdot) Q(\cdot) \right)}}{\sqrt{1 + (1 + \frac{1}{\epsilon^*}) \delta^2 Q(\cdot)}} \right\|_\infty^2,$$

(10)

where

$$\epsilon^* = \arg \min_{\epsilon > 0} \inf_{Q(\cdot) \in H^\infty} \left\| \frac{\sqrt{1 + \epsilon \left( 1 - P(\cdot) Q(\cdot) \right)}}{\sqrt{1 + (1 + \frac{1}{\epsilon}) \delta^2 Q(\cdot)}} \right\|_\infty^2.$$  

(11)

In particular, when $p_{\text{min}} < 2$, we have

$$\epsilon^* = \frac{\delta (p_{\text{min}} - \delta)}{1 - \delta (p_{\text{min}} - \delta)},$$

(12)

and the optimal $H^\infty$ norm becomes

$$\gamma_{\text{opt}} = \frac{1}{1 + (p_{\text{min}} - \delta)^2},$$

(13)

which is the same as that obtained from max-min problem:

$$\sup_{\|\Delta P(\cdot)\|_\infty \leq \delta} \inf_{Q(\cdot) \in H^\infty} \left\| \left[ 1 - \left( P(\cdot) + \Delta P(\cdot) \right) Q(\cdot) \right] / Q(\cdot) \right\|_\infty.$$  

(14)

Thus the method presented here for solving robust two-block problems always works for the tracking problem. Moreover, it is interesting that for $p_{\text{min}} < 2$ the solutions to the min-max problem (6) and the max-min problem (14) coincide. [In general we have min-max $\geq$ max-min.]

**References**